Physics, Measurements, and Numerical Methods

in an Eulerian-Lagrangian Reference Frame

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Coastal Ocean Modeling

Acknowledgement and Background

Physics -- Fluid Dynamics is Lagrangian by nature

Eulerian treatments are for convenience

Measurements based on a Lagrangian point of view

- Dye studies, drifters and all that
- PIV in laboratories
- What can we do for coastal oceans?

Numerical methods

- Eulerian-Lagrangian formulation

Lagrangian Residual Currents and

- Long-Term Transport

Conclusions and Recommendation

Joseph Louis Lagrange

(Giuseppe Luigi Lagrangia)

1736-1813

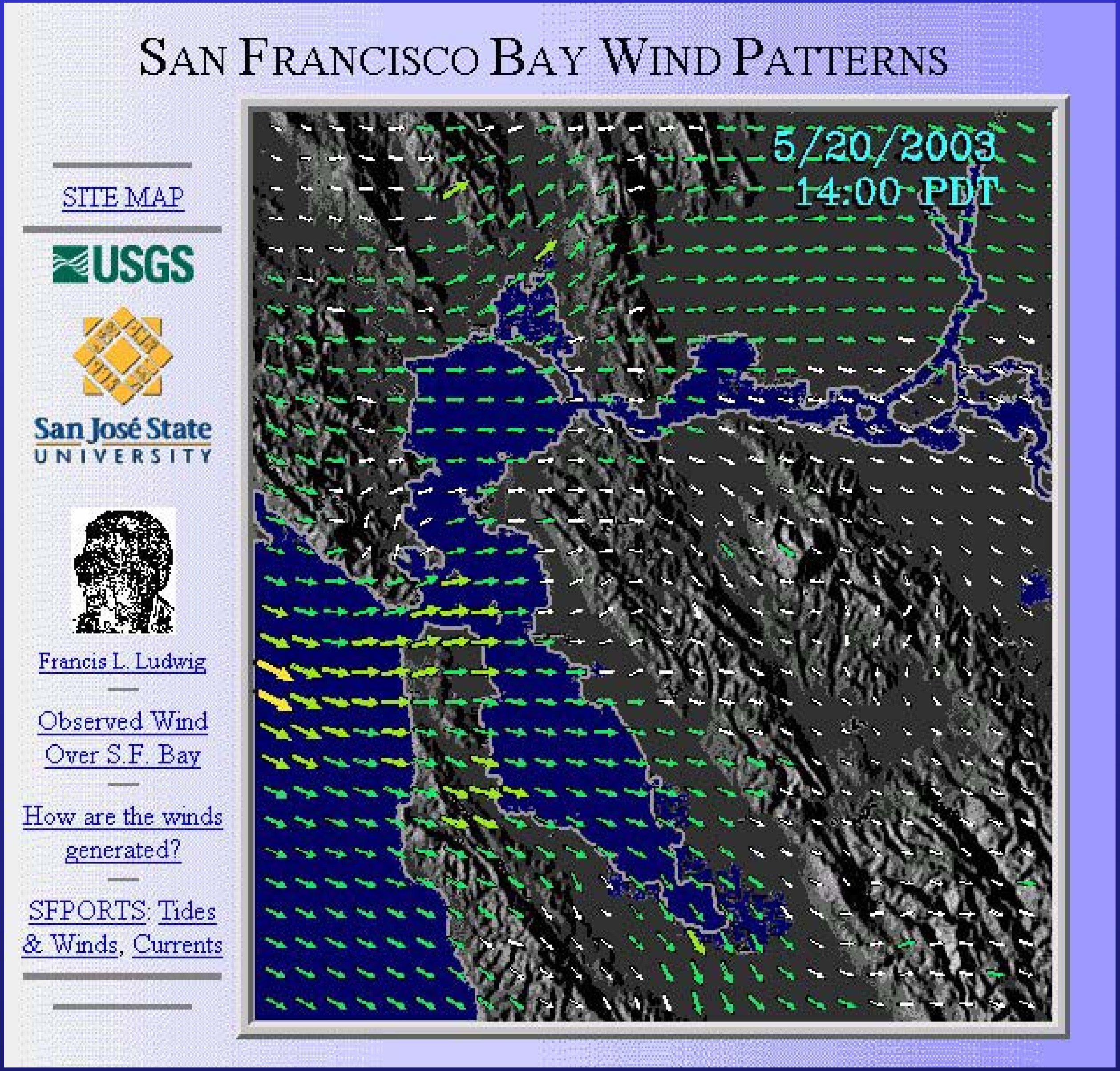
1766: Frederick the Great (Berlin) recruited him to take the position vacated by Euler, as the court mathematician

1787: Louis XVI invited him to Paris

Mechanique Analytique:

To unite and present from one point of view the different principles in mechanics

Eulerian Representation



Eulerian Variable: $\theta = \theta(x, y, z, t)$

Lagrangian Representation

Lagrangian Variable: $\theta = \theta[\vec{X}_o(t_o), \vec{X}(t), t]$

Physics

Forward Problem:	Inverse Problem:
Search and rescue	Search for evidences

Lagrangian	vs.	Eulerian
Discrete		Continuum
Spilled Oil Slicks		Dissolved Solutes
Sediment Patches		Pollutants
Planktons and Larvae		Salt, Temperature
(Biology)		

Physics

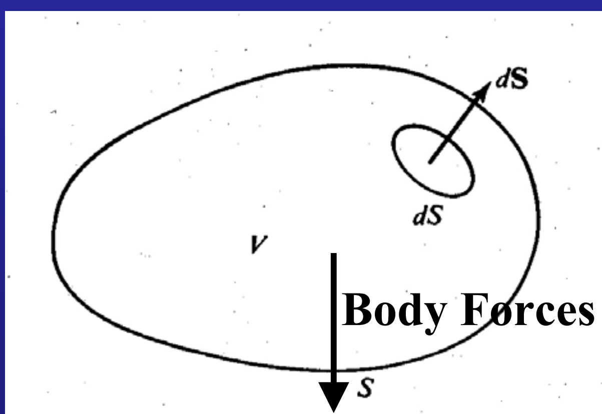
Kinematics

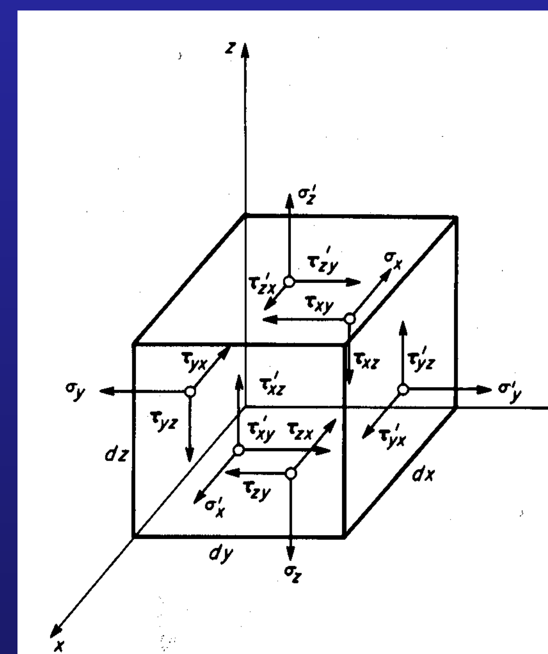
Second Law of Newton
In Fluid Dynamics

Lagrangian P.V:

$$\vec{F} = m\vec{a}$$

Eulerian P.V:





$$\sum \textbf{Surface and Body Forces} = \frac{D}{Dt}(Momemtum)$$

Observations:

Lagrangian Point of View:

- Physics is clear
- Discrete particle dynamics
- Measurement difficulties
- Hard to quantify measurements

Eulerian Point of View:

- Continuum
- Operational Convenience
- Easy to organize "information"

Substantial Derivative: Euler-Lagrangian Transformation

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z}$$

Some Common Measurement Techniques:

Eulerian Reference Frame:

Eulerian Variable: $\boxed{\theta = \theta(x, y, z, t)}$

Fixed Current Meter, CTD moorings

Cruising and Profiling ADCP, CTD

HF Radar for surface current and waves

Operational Convenience, Easy to organize "information"

Lagrangian Reference Frame:

Lagrangian Variable: $\boxed{\theta = \theta[\vec{X}_o(t_o), \vec{X}(t), t]}$

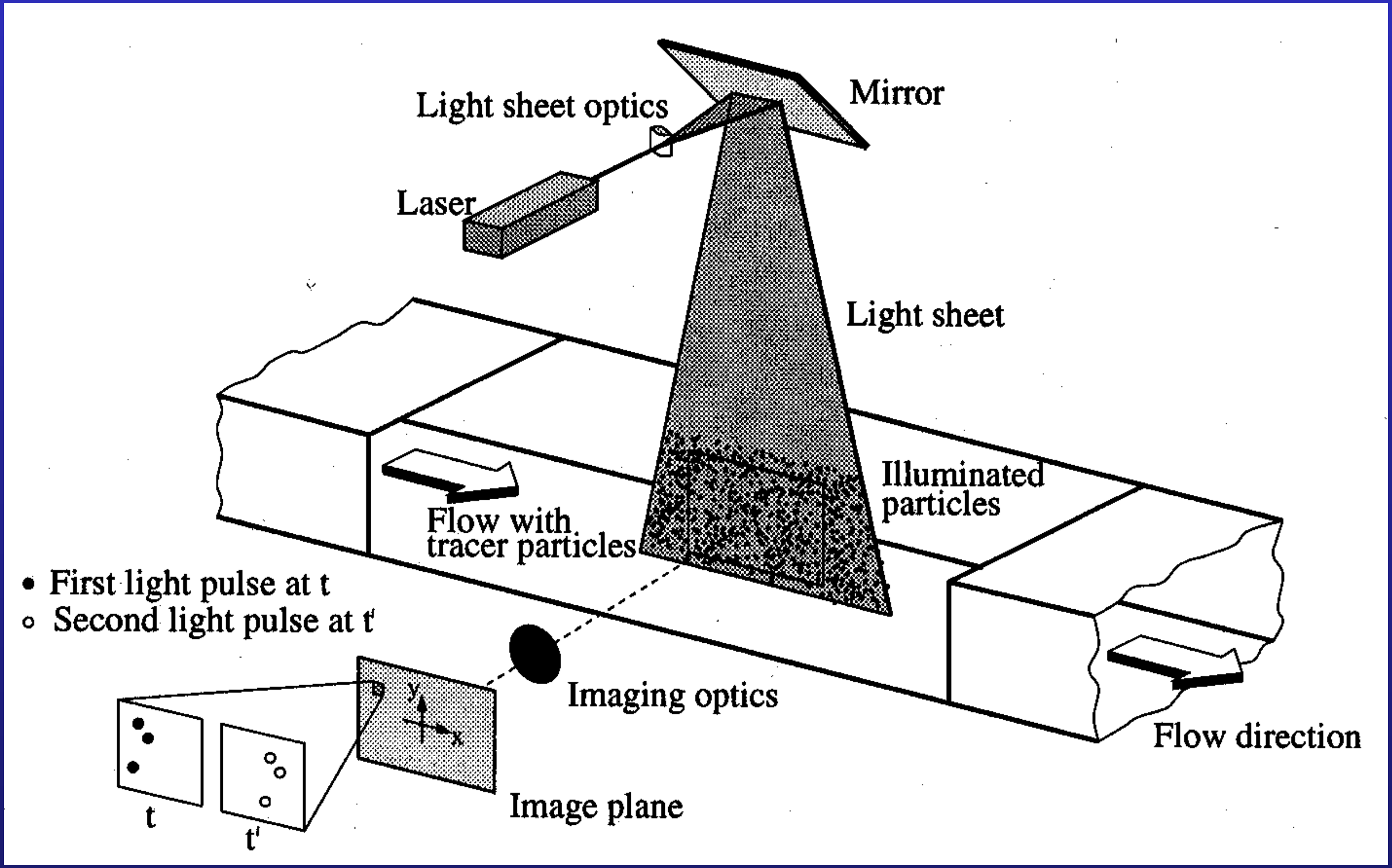
Most flow visualization techniques

Dye studies, drifters

Long-term path of water 'mass'

Measurement Difficulties, Hard to quantify measurements

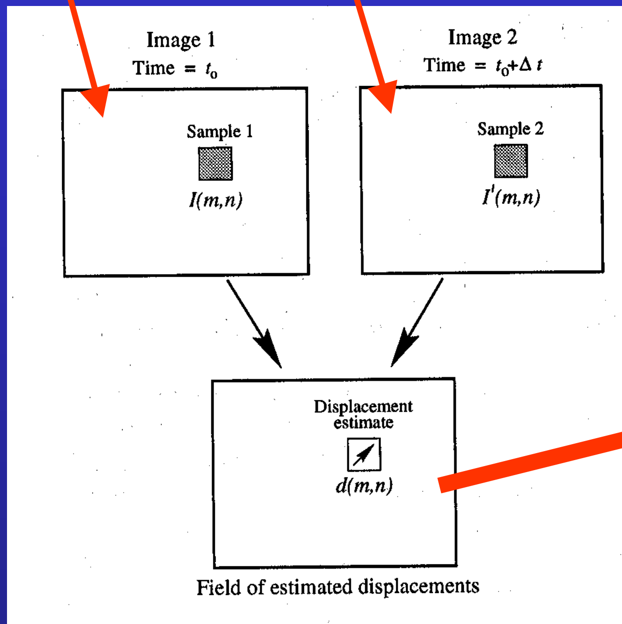
Combined Eulerian-Lagrangian Measurement Techniques: Particle Image Velocimetry (PIV)

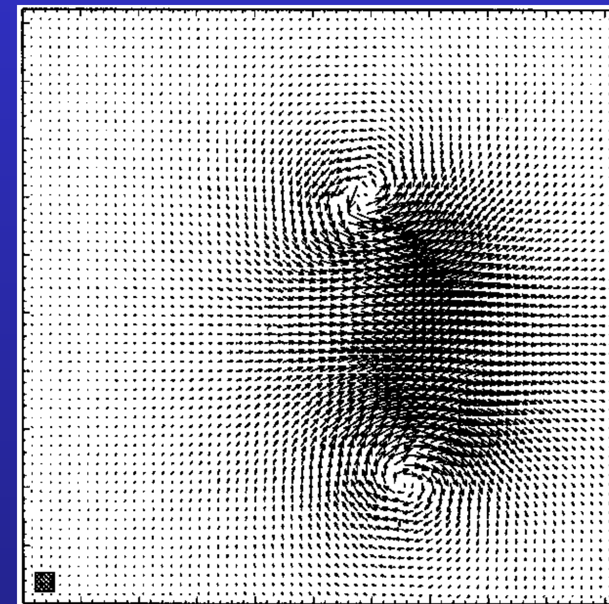


<u>Particle Image Velocimetry</u> by M. Raffel, C. Willert, J. Kompenhans, Springer, 1998.

Lagrangian Observations

Map results to an Eulerian Reference Frame





Estimating displacements by cross-correlations

Combined Eulerian-Lagrangian Measurement Techniques

PIV has been successfully extended to include multi-cameras, to three-dimensional flows, turbulence, …., etc.

Observation: The technique is mature in lab applications!

Are there rooms for applications of Particle Image Velocimetry (PIV) in geophysical fluid flows?

Have you noticed that weather forecasts are more accurate?

Difference? Temporal and spatial scales, Tracers

Some applications in rivers

We have limited success in field applications

Challenge #1: Does PIV have any potential in coastal ocean studies?

Numerical Methods

Lagrangian Point of View:

Clear Physics

Difficulties to quantify measurements

Eulerian Point of View:

Continuum, Operational Convenience

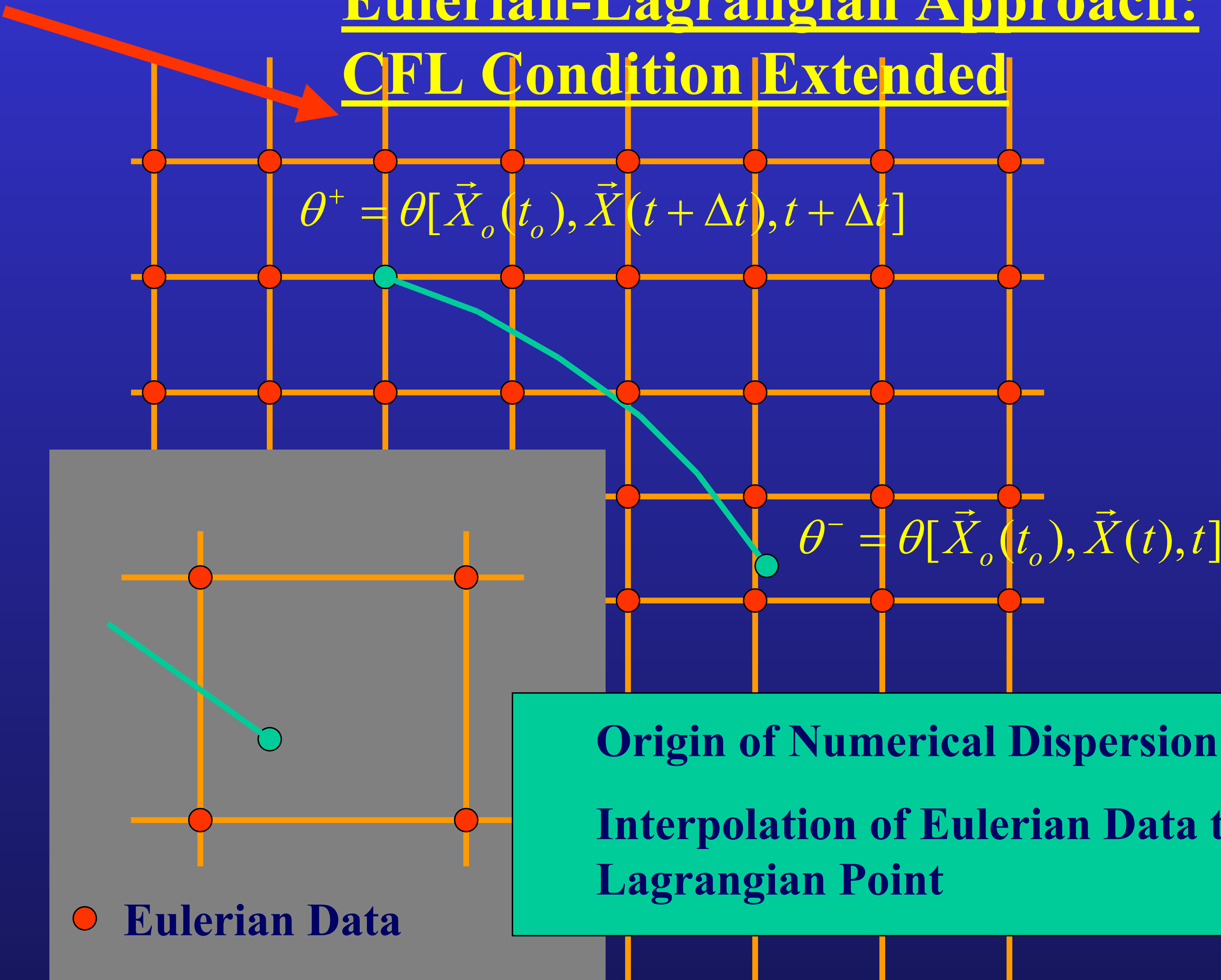
Easy to organize "information"

Substantial Derivative: Euler-Lagrangian Transformation

$$\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + w\frac{\partial\theta}{\partial z}$$

$$\frac{D\theta}{Dt} = \frac{\theta^{+} - \theta^{-}}{\Delta t} = \frac{\theta[X(t_o + \Delta t)] - \theta[X(t_0)]}{\Delta t}$$

Eulerian-Lagrangian Approach: CFL Condition Extended



Origin of Numerical Dispersion:

Interpolation of Eulerian Data to Lagrangian Point

TRIM family of models:

Casulli, V., 1990, Semi-implicit Finite-difference Methods for the Two-dimensional Shallow Water Equations, J. Comput. Phys., V. 86, p. 56-74.

Stability Analysis: Gravity wave terms and velocities in Continuity Eq. control the numerical stability

Method of Solution:

1. Treat those terms implicitly, and the remaining terms explicitly.
2. Substituting momentum Eqs. into continuity Eq., resulting a matrix equation that determines the water surface of the entire domain.

TRIM_2D: Extensive applications in San Francisco Bay

Cheng, R. T., V. Casulli, and J. W. Gartner, 1993, Tidal, residual, intertidal mudflat (TRIM) model and its applications to San Francisco Bay, California, Estuarine, Coastal, and Shelf Science, Vol. 36, p. 235-280.

2D Depth-Averaged Shallow Water Equations

Continuity Eq.:

$$\frac{\partial \varsigma}{\partial t} + \frac{\partial[(h+\varsigma)U]}{\partial x} + \frac{\partial[(h+\varsigma)V]}{\partial y} = 0$$

X-Momentum Eq.:

$$\frac{DU}{Dt} - fV = -g\frac{\partial \varsigma}{\partial x} + \frac{1}{\rho_o(h+\zeta)}(\tau_x^w - \tau_x^b) + A_h \nabla^2 \mathrm{U} - \frac{g}{2\rho_o}(h+\varsigma)\frac{\partial \rho}{\partial x}$$

Y-Momentum Eq.:

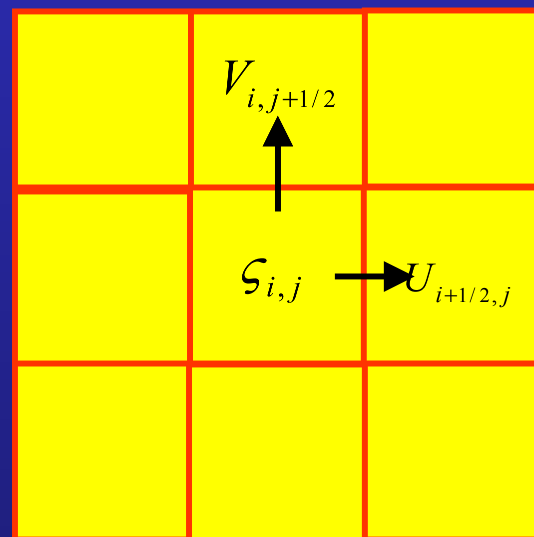
$$\frac{DV}{Dt} + fU = -g\frac{\partial \varsigma}{\partial y} + \frac{1}{\rho_o(h+\zeta)}(\tau_y^w - \tau_y^b) + A_h \nabla^2 \mathrm{V} - \frac{g}{2\rho_o}(h+\varsigma)\frac{\partial \rho}{\partial y}$$

Eulerian-Lagrangian Method (ELM) => Stability (von Neumann)

X-Momentum Eq.:

$$\frac{DU}{Dt} - fV = -g\frac{\partial \varsigma}{\partial x} + \frac{1}{\rho_o(h+\zeta)}(\tau_x^w - \tau_x^b) + A_h \nabla^2 \mathrm{U} - \frac{g}{2\rho_o}(h+\varsigma)\frac{\partial \rho}{\partial x}$$

Semi-implicit FD: Algebraic Eq. of $\varsigma_{i,j}^{n+1}, U_{i+1/2,j}^{n+1}, \varsigma_{i+1,j}^{n+1}$



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Semi-implicit FD: Algebraic Eq. of $\varsigma_{i,j}^{n+1}, V_{i,j+1/2}^{n+1}, \varsigma_{i,j+1}^{n+1}$

Substituting the momemtum Equations into

Continuity Eq.: $$\frac{\partial \varsigma}{\partial t} + \frac{\partial[(h+\varsigma)U]}{\partial x} + \frac{\partial[(h+\varsigma)V]}{\partial y} = 0$$

$$(1 + A_{i+1,j} + B_{i-1,j} + C_{i,j+1} + D_{i,j-1})\varsigma_{i,j}^{n+1}$$
$$- A_{i+1,j}\varsigma_{i+1,j}^{n+1} - B_{i-1,j}\varsigma_{i-1,j}^{n+1} - C_{i,j+1}\varsigma_{i,j+1}^{n+1} - D_{i,j-1}\varsigma_{i,j-1}^{n+1} = E_{i,j}^{n}$$

With all coefficients are positive.

The governing matrix equation is symmetric, diagonally dominant, and positive definite. Numerical solution is achieved by a preconditioned conjugate gradient method.

Some Numerical Properties

- Eulerian-Lagrangian method is used for D[ ]/Dt
- Implicit terms - unconditionally stable (von Neumann sense)
- Discretized equation - properly accounts for positive and zero depths
- Wetting and drying of cells are treated correctly
- Pentadiagonal solution matrix - solved efficiently by preconditioned conjugate gradient method
- The model is robust and efficient

Systematic Development of TRIM Models:

TRIM_3D: Applications in San Francisco Bay and others

Casulli, V. and R. T. Cheng, 1992, Inter. J. for Numer. Methods in Fluids

Casulli, V. and E. Cattani, 1994, Comput. Math. Appl., Stability, accuracy and efficiency analysis of TRIM_3D, θ-method for time-difference

Cheng, R. T. and V. Casulli, 1996, Modeling the Periodic Stratification and Gravitational Circulation in San Francisco Bay, ECM-4.

TRIM_3D: Non-hydrostatic

Casulli, V. and G. S. Stelling, 1996, ECM-4

Casulli, V. and G. S. Stelling, 1998, ASCE, J. of Hydr. Eng

UnTRIM model:

Casulli, V. and P. Zanolli, 1998, A Three-dimensional Semi-implicit Algorithm for Environmental Flows on Unstructured Grids, Proc. of Conf. On Num. Methods for Fluid Dynamics, University of Oxford.

Extension to Unstructured Grid Model -- UnTRIM

TRIM Modeling Philosophy:

1. **Semi-implicit Finite-Difference Methods**
2. **Θ-Method for time difference**
3. **Solutions in Physical Space, regular mesh, no coordinate transformations in x-, y-, or z-directions**
4. **In complicated domain, refine grid resolution if necessary**
5. **Pursue computational efficiency and robustness**

UnTRIM (Unstructured Grid TRIM model) follows the SAME TRIM modeling philosophy, except the finite-difference cells are boundary fitting unstructured polygons!

Summary of Numerical Algorithm

Governing equations (Hydrostatic Assumption)

Continuity and Free-surface Equations

$$Div(\vec{U}) = 0$$

Incompressibility

$$\frac{\partial}{\partial}\frac{\eta}{t} + \nabla \bullet \left[\int_{-h}^{\eta} \vec{V}\, dz \right] = 0$$

Free-surface equation

Horizontal Momentum Equation in $\vec{N}_j$ direction for velocity V_j

$$\frac{DV_j}{Dt} - f(\nabla \times \vec{V}) \bullet \vec{N}_j = \frac{\partial}{\partial z}\left(\nu_v \frac{\partial}{\partial z} V_j\right) + \nu_h \nabla^2 V_j - g\frac{\partial}{\partial}\frac{\eta}{N_j} - \frac{g}{\rho_o}\frac{\partial}{\partial N_j}\int_z^{\eta}(\rho - \rho_o)\, dz$$

where $\nabla\times(\ )$ is cross product, $\nabla\bullet(\ )$ is inner product, $\nabla^2(\ )$ is the Laplacian, and $\vec{V}$ is the velocity in the horizontal plane.

Transport Equations

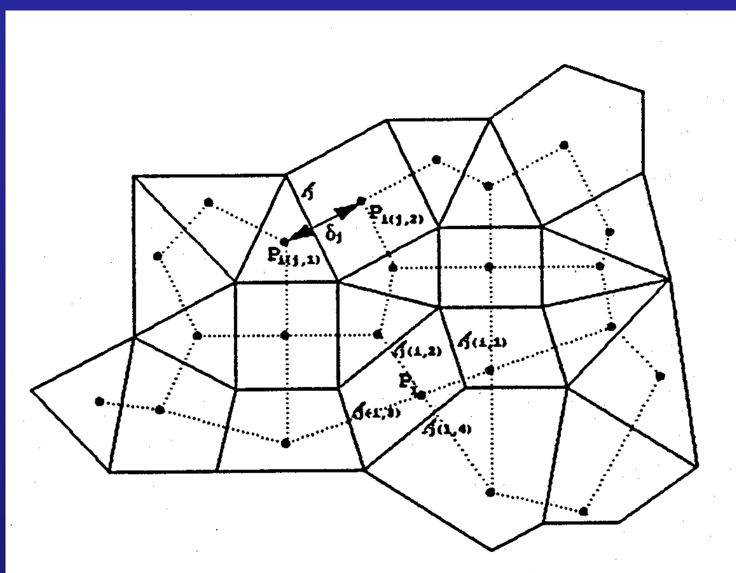
$$\frac{D}{Dt}\mathbf{C}_j = \frac{\partial}{\partial z}\left(\mathbf{K}_v \frac{\partial}{\partial z}\mathbf{C}_j\right) + \mathbf{K}_h \nabla^2 \mathbf{C}_j \qquad \mathbf{j = 1, 2, 3, \ldots}$$

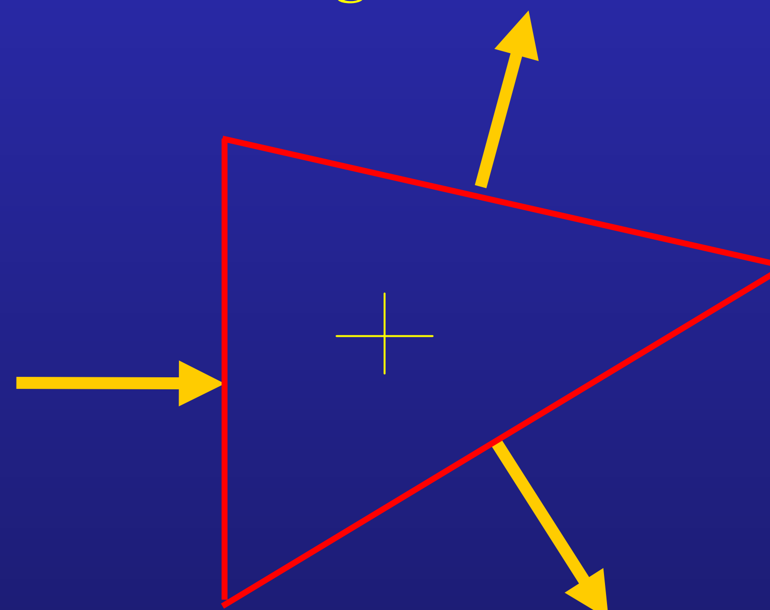
Lagged one time-step

And an equation of State

1. Semi-implicit finite-difference of momentum Eq. in the normal direction to each face is applied!

2. Applied the Finite-Volume integration of the free surface equation!
   Local and global conservation of volume is guaranteed!





3. The resultant matrix equation determines the water surface elevation for the entire field.

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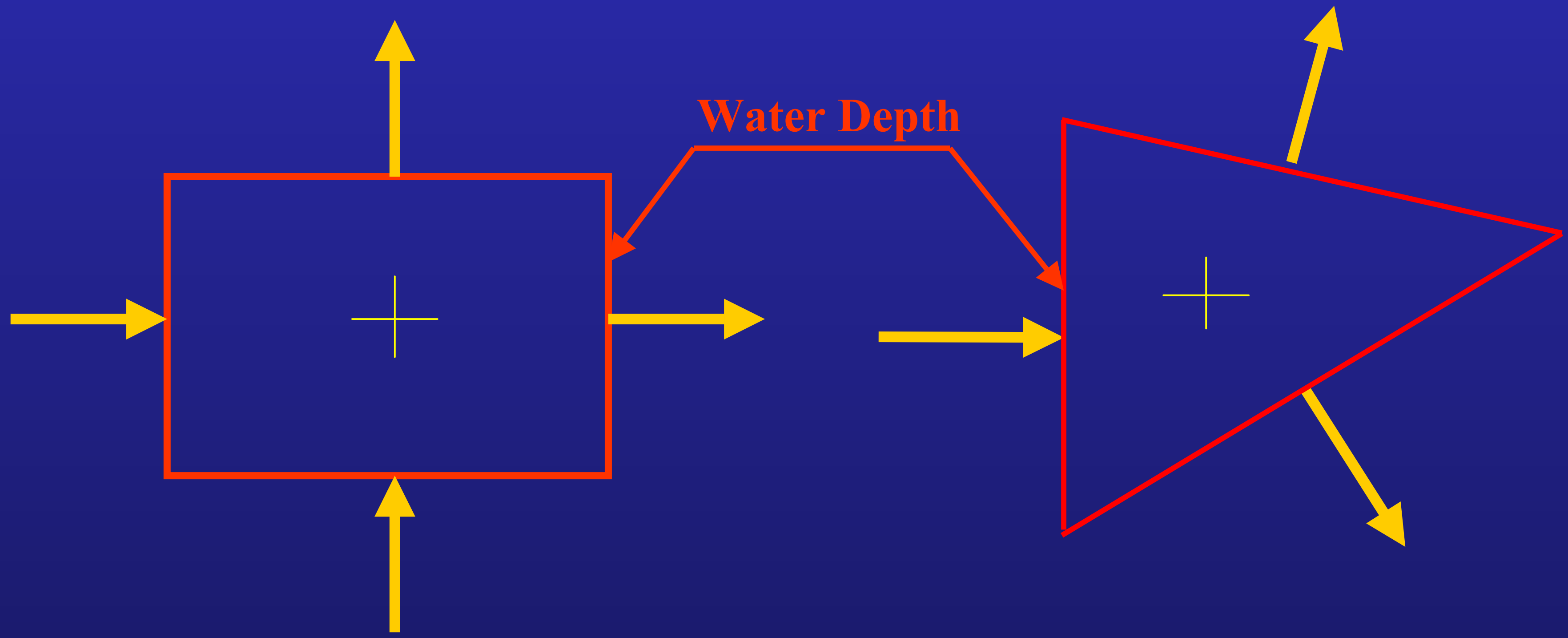


3. The resultant matrix equation determines the water surface elevation for the entire field.

Summary of Numerical Algorithm

Momentum Equation in $\vec{N}_j$ direction for velocity V_j relates

V_j and η (left) and η (right) on each face of a polygon

Continuity and Free-surface Equations

$$Div(\vec{U}) = 0$$

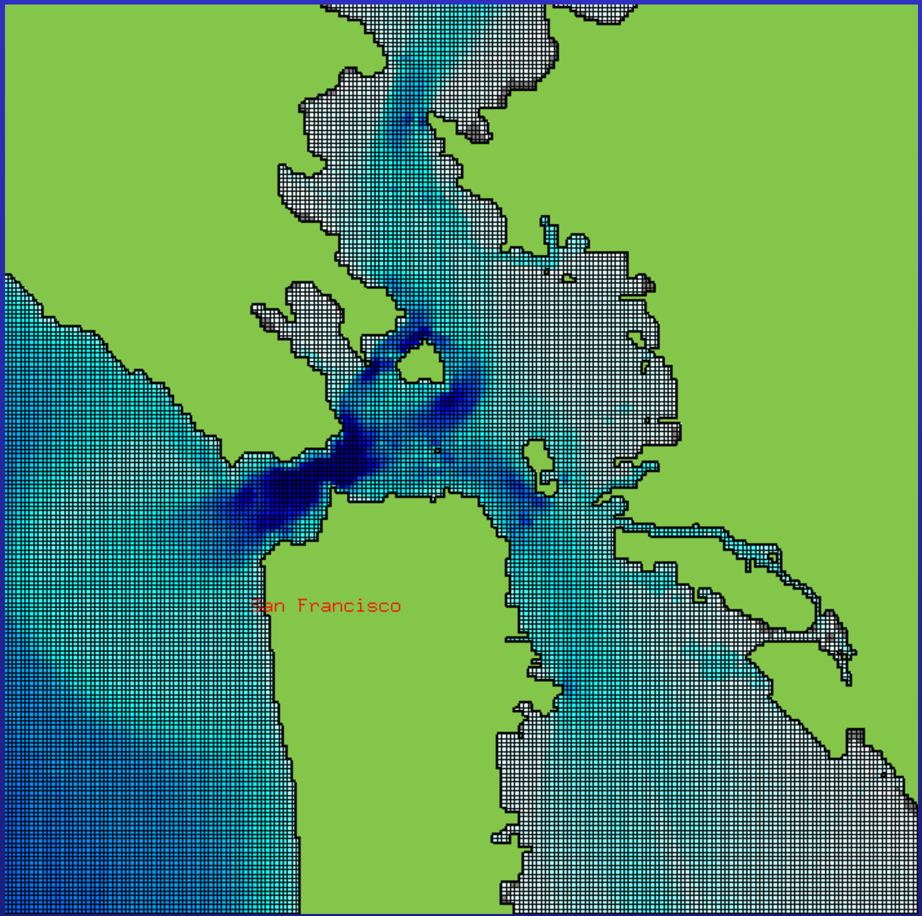
$$\frac{\partial \eta}{\partial t} + \nabla \bullet \left[\int_{-h}^{\eta} \vec{V}\, dz \right] = 0 \qquad \Rightarrow \qquad \frac{\partial \eta}{\partial t} + \oint \left(\int_{-h}^{\eta} \vec{V} dz \right) \bullet d\vec{s} = 0$$

Finite Volume integration over each polygon =>
V's are eliminated giving a Matrix Eq. for η

The continuity equation and the momentum equations are truly coupled in the solution. No mode splitting is used!

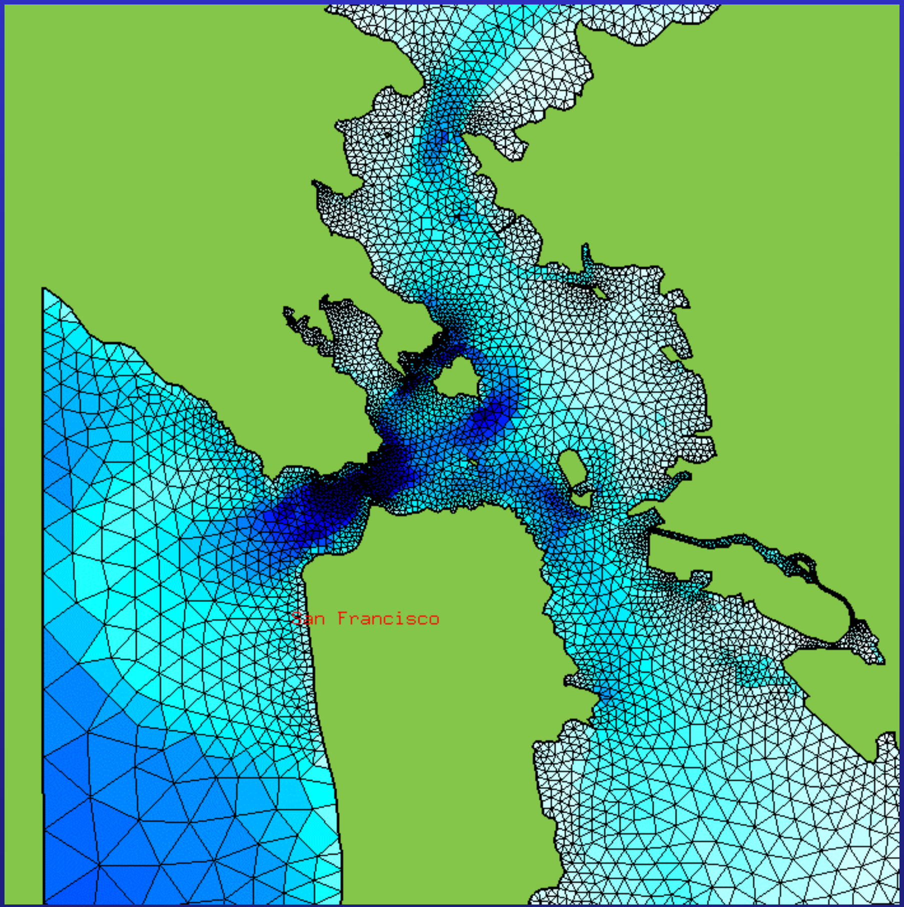
San Francisco Bay, California

(All Rectangles)



48506 nodes, 45841 polygons
94374 sides on the top layer
42 layers, 1,160 K faces, Δt = 180
72 hours simulation requires 4.06 hours (R= 17.7) CPU on 2.2 GHz PC

(Mixed Polygons)



12682 nodes, 20126 polygons
32827 sides on the top layer
42 layers, 295 K faces, Δt = 180
72 hours simulation requires 1.03 hours (R= 70) CPU on 2.2 GHz PC

Numerical Model is an Eulerian Database
Lagrangian Numerical Experiments

Eulerian-Lagrangian Collaboration

Lagrangian Point of View:

- Clear Physics
- Discrete Labeled Water Parcel
- Measurement Difficulties (Easier numerically)
- Hard to quantify measurements (We will see!)

Eulerian Point of View:

- Operational Convenience
- Easy to organize “information”
- Needed “information” are populated on
  an Eulerian Model Grid points (database)

Long-term Transport and Residual Currents

Eulerian Residual Current:

$$\vec{V}_{er}(\vec{X}_o) = \frac{1}{T}\int_{t_o}^{t_o+T} \vec{V}(\vec{X}_o, t')dt'$$

$$\vec{V}_{er}(\vec{X}_o) = \frac{1}{T}\sum_{i=0}^{N} \vec{V}_e(\vec{X}_o, i\Delta t)\Delta t$$

$$\vec{V}_{er}(\vec{X}_o) = \frac{Y(t_o + T) - X(t_o)}{T}$$

Progressive Vector Diagram

Lagrangian Residual Current:

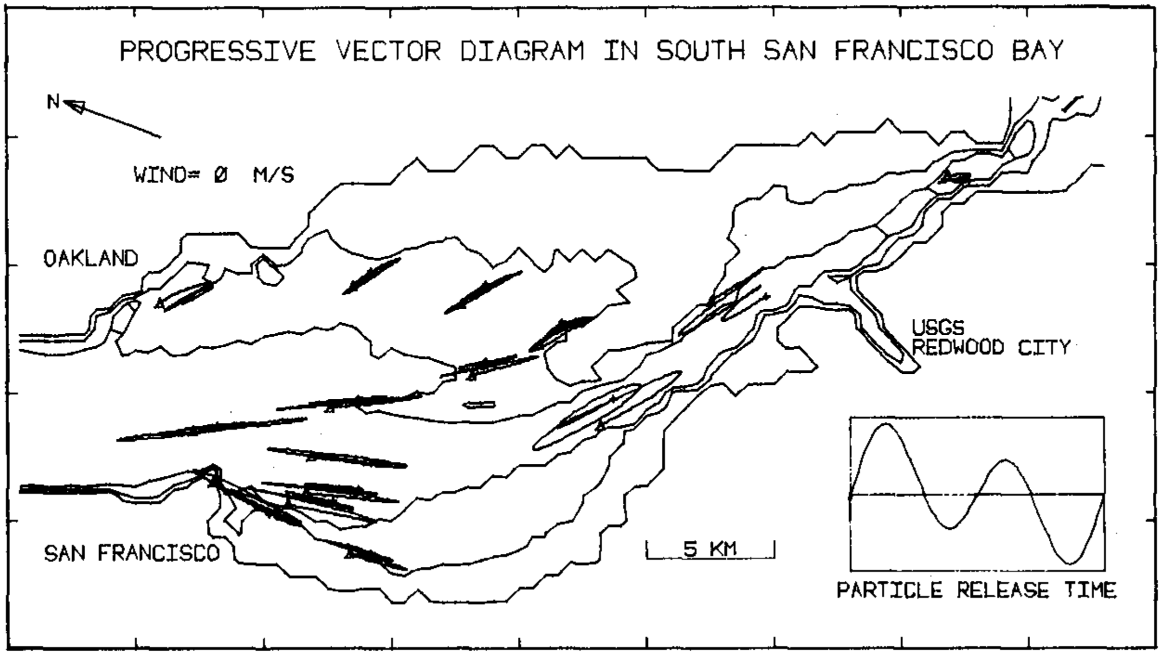
$$\vec{V}_{lr}(\vec{X}_o, t_o) = \frac{1}{T}\int_{t_o}^{t_o+T} \vec{V}[\vec{X}(t'), t']dt'$$

$$\vec{V}_{lr}(\vec{X}_o) = \frac{1}{T}\sum_{i=0}^{N} \vec{V}_l(\vec{X}, i\Delta t)\Delta t$$

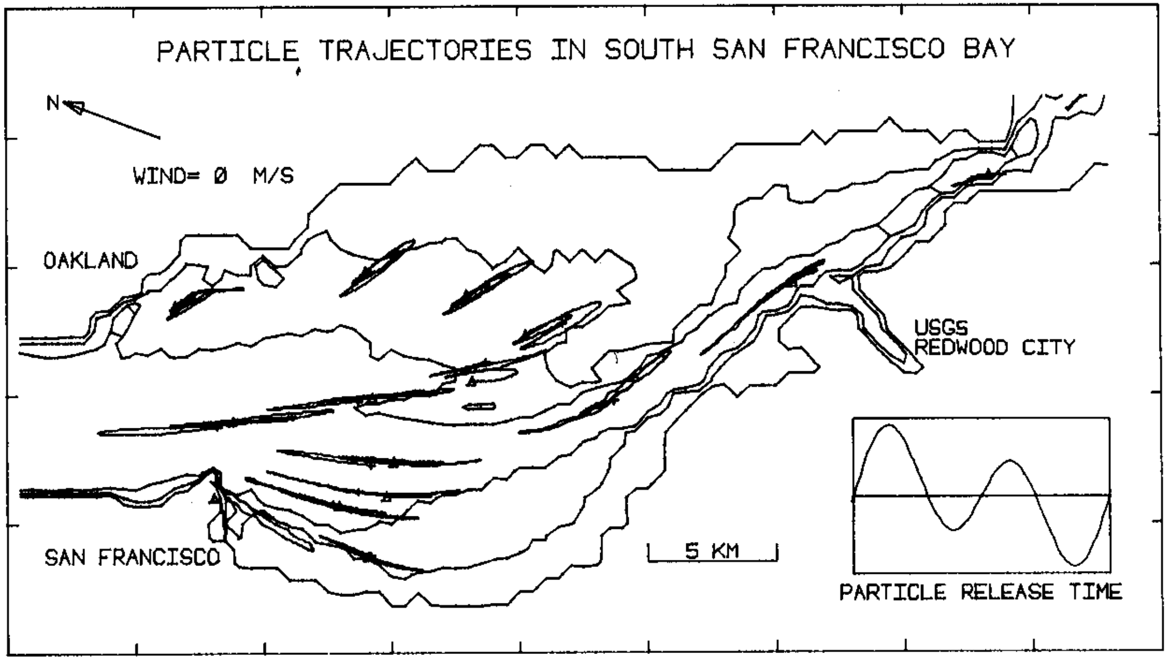
$$\vec{V}_{lr}(\vec{X}_o, t_o) = \frac{X(t_o + T) - X_o(t_o)}{T}$$

Labeled Water Parcel Trajectory

Progressive Vector Diagram



Eulerian Residual Current:



Water Parcel Trajectory

Lagrangian Residual Current:

Long-term Transport and Residual Currents

Eulerian Residual Current:

$$\vec{V}_{er}(\vec{X}_o) = \frac{1}{T}\int_{t_o}^{t_o+T} \vec{V}(\vec{X}_o, t')dt'$$

Lagrangian Residual Current:

$$\vec{V}_{lr}(\vec{X}_o, t_o) = \frac{1}{T}\int_{t_o}^{t_o+T} \vec{V}[\vec{X}(t'), t']dt'$$

Weakly Nonlinear System

$$x = x_o + \kappa\xi; \quad y = y_o + \kappa\eta; \quad k = u_c / \sqrt{gh} = u_r / u_c = \zeta_c / h_c$$

$$\vec{V}_l(x, y, t) = \vec{V}_e(x_o, y_o, t) + \kappa[(\partial\vec{V}/\partial x)_o\xi + (\partial\vec{V}/\partial y)_o\eta]$$

$$+\kappa^2[-----] + O(\kappa^3)$$

Results of Weakly Nonlinear Small Perturbation Analysis:

$$\vec{V}_{lr}(x_o, y_o, t_o) = \vec{V}_{er}(x_o, y_o) + \vec{V}_{sd}(x_o, y_o) + \kappa \vec{V}_{ld}(x_o, y_o, t_o)$$

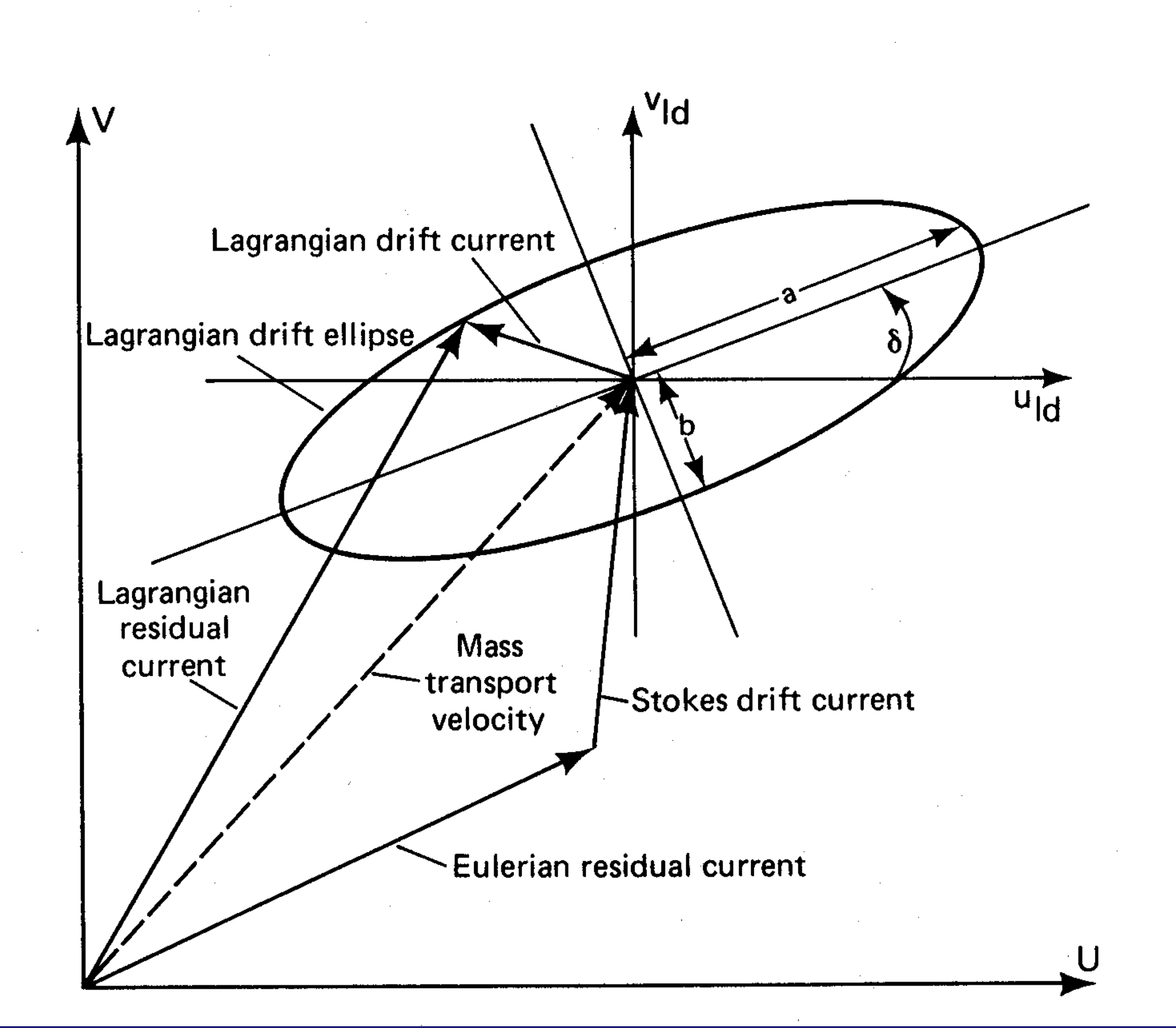
Lagrangian Residual Current $=$ Eulerian Residual Current $+$ Stokes Drift $+\kappa$ Lagrangian Drift

Velocity Gradient

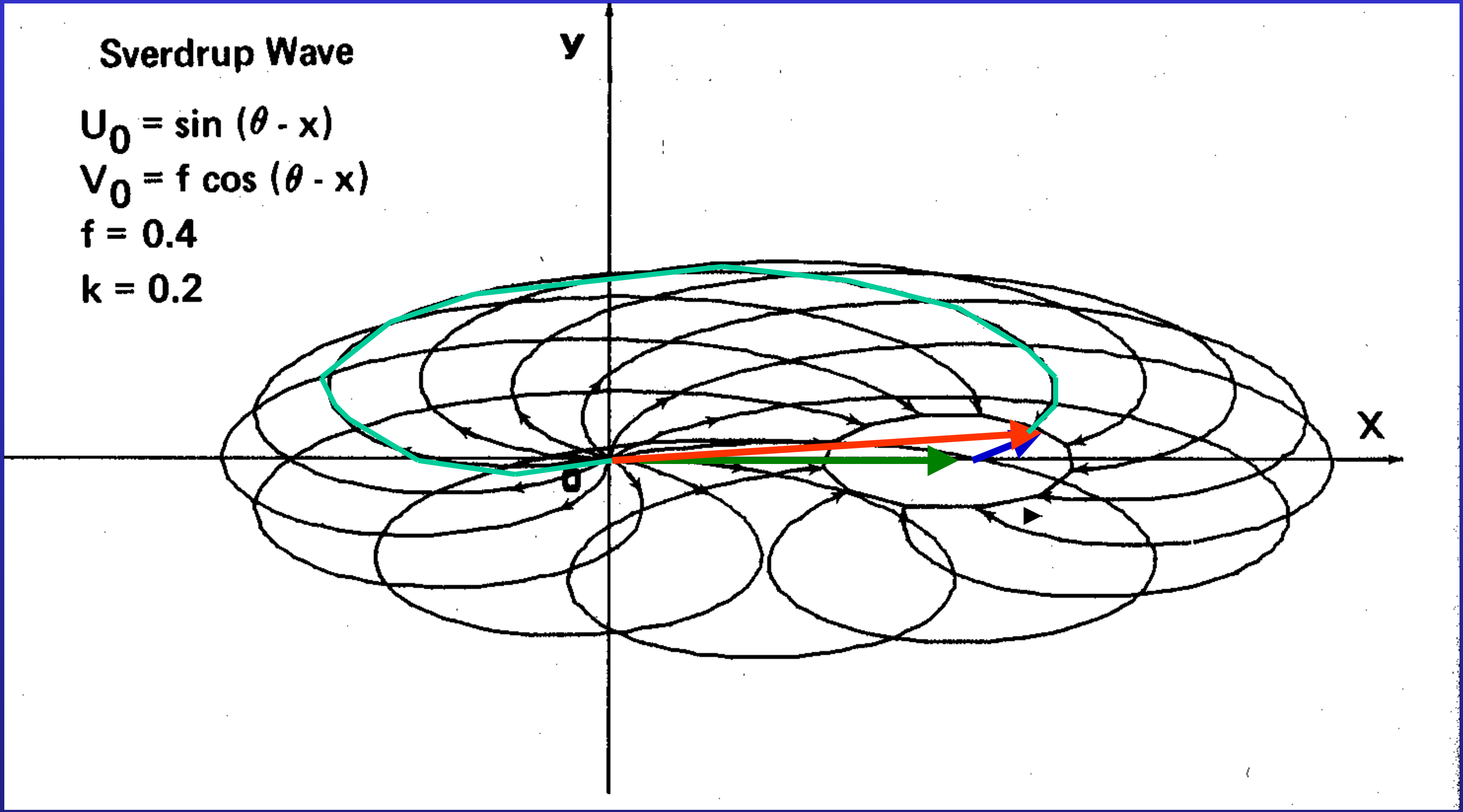
Stress: Second derivatives of velocity

Longuet-Higgins (1969)
Zimmerman (1979)

Mass Transport Velocity $=$ Eulerian Residual Current $+$ Stokes Drift



Lagrangian Residual Current and Lagrangian Drift



$$\vec{V}_{lr}(x_o, y_o, \theta_o) = \vec{V}_{er}(x_o, y_o) + \vec{V}_{sd}(x_o, y_o) + \kappa \vec{V}_{ld}(x_o, y_o, \theta_o)$$

$$\vec{V}_{er}(x_o, y_o) = 0 \qquad u_{ld}(x_o, y_o, \theta_o) = 1/2\sin(\theta_o - x_o - \pi)$$

$$\vec{V}_{sd}(x_o, y_o) = 1/2 \qquad v_{ld}(x_o, y_o, \theta_o) = f/2\cos(\theta_o - x_o - \pi)$$

Long-term Transport and Residual Currents

Two Pathways to Long-term Transport:

1. **Direct integration of transport equation**

$$\frac{\partial \theta}{\partial t} + \vec{V} \bullet \nabla \theta = D \nabla^2 \theta$$

2. **Seek for an intertidal transport equation**

$$\frac{\partial < \theta >}{\partial t} + \vec{V}_{er} \bullet \nabla < \theta > = < \vec{V}' \bullet \nabla \theta' > + < D \nabla^2 \theta >$$

where <…> is tidally averaged

$$\vec{V} = \vec{V}_{er} + \vec{V}'(t) \qquad \theta = < \theta > + \theta'(t)$$

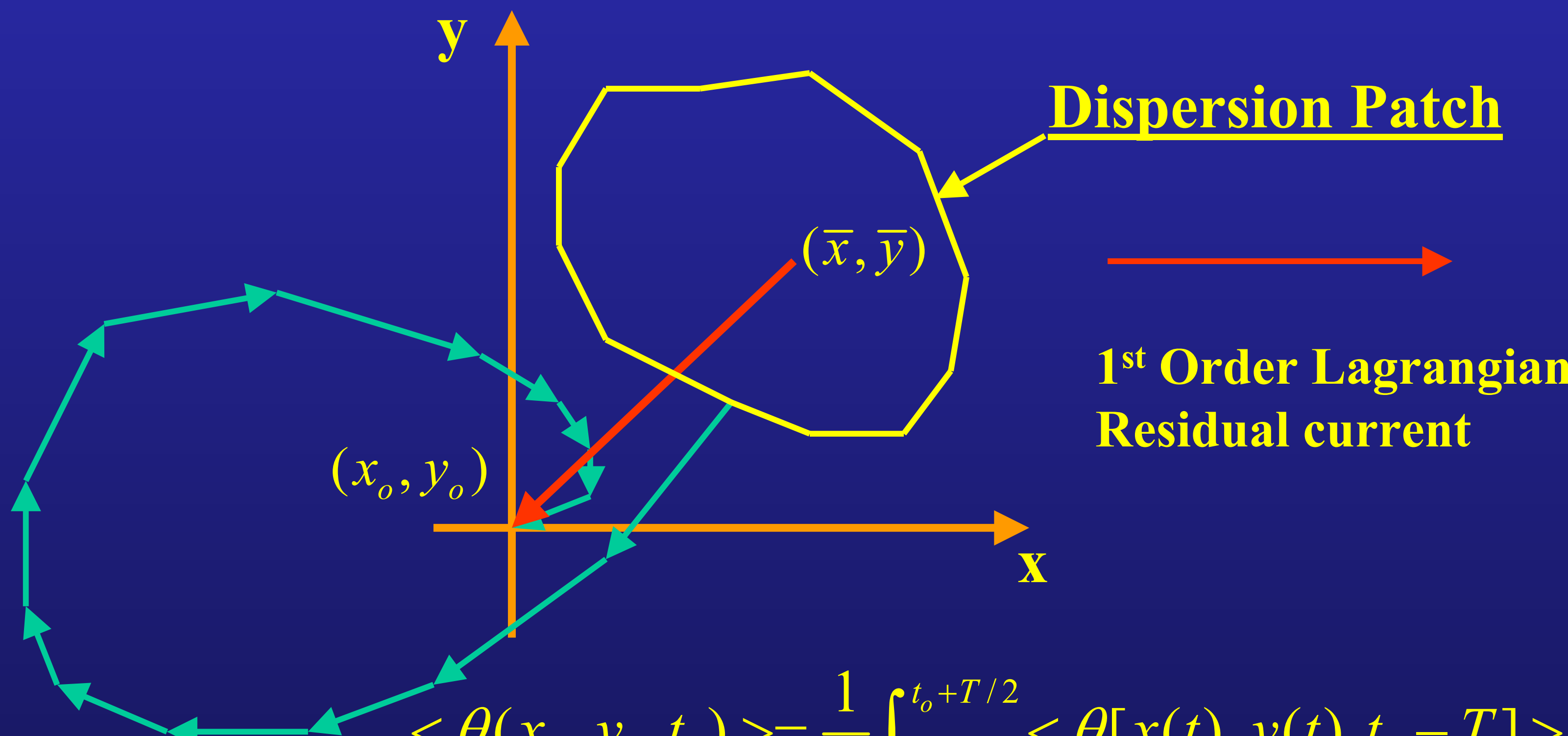
3. **Small Perturbation Analysis:**

$$(\vec{V}_{er} + \vec{V}_{sd}) \bullet \nabla < \theta > = \kappa (Dispersion)$$

Mass Transport Velocity $< \vec{V}' \bullet \nabla \theta' > = \vec{V}_{sd} \bullet \nabla < \theta >$

Generalized Intertidal Transport Equation and Tidal Dispersion

Time Average vs. Ensemble Average
(Eulerian) (Lagrangian)



$$< \theta(x_o, y_o, t_o) > = \frac{1}{T}\int_{t_o - T/2}^{t_o + T/2} < \theta[x(t), y(t), t_o - T] > dt$$

Generalized Intertidal Transport Equation a

$$\frac{<\theta(x_o, y_o, t_o)> - <\theta(\bar{x}, \bar{y}, t_o - T)>}{T} =$$

$$\frac{\frac{1}{T}\int_{t_o - T/2}^{t_o + T/2} <\theta[x(t), y(t), t_o - T]> dt - <\theta[\bar{x}, \bar{y}, t_o - T]>}{T}$$

Taylor Series Expansion about $(\bar{x}, \bar{y}, t_o - T)$

$$\frac{\partial <\theta>}{\partial t} + <\vec{V}_{lr}> \bullet \nabla <\theta> =$$

$$D_{xx}\frac{\partial^2 <\theta>}{\partial x^2} + 2D_{xy}\frac{\partial^2 <\theta>}{\partial x \partial y} + D_{yy}\frac{\partial^2 <\theta>}{\partial y^2}$$

$$\frac{\partial <\theta>}{\partial t} + <\vec{V}_{lr}> \bullet \nabla <\theta> = D_{xx}\frac{\partial^2 <\theta>}{\partial x^2} + 2D_{xy}\frac{\partial^2 <\theta>}{\partial x \partial y} + D_{yy}\frac{\partial^2 <\theta>}{\partial y^2}$$

where

$$<\vec{V}_{lr}> = \frac{1}{T}[(\overline{x} - x_o), (\overline{y} - y_o)]$$

$$D_{xx} = \frac{1}{2T^2}\int_{t_o - T/2}^{t_o + T/2} [x(t) - \overline{x}]^2 dt$$

$$D_{xy} = \frac{1}{2T^2}\int_{t_o - T/2}^{t_o + T/2} [x(t) - \overline{x}][y(t) - \overline{y}] dt$$

$$D_{yy} = \frac{1}{2T^2}\int_{t_o - T/2}^{t_o + T/2} [y(t) - \overline{y}]^2 dt$$

These results gives clear physics, consistent to weakly nonlinear analysis without invoking weakly nonlinear approximation

To define ‘dispersion patch’, the hydrodynamic equations need to be integrated ‘backward’ in time

Computations of tidal dispersion coefficients for San Francisco Bay show correct order of magnitude

Challenge #2: How do we validate these computations and implement this approach for practical applications?

Conclusion:

Lagrangian VP gives clear Physics but difficult to Manage!

Eulerian VP is well suited for quantification!

Recommendation:

Think as a Lagrangian!

Act as an Eulerian!

Thank you!